# Chapter 2
# How Mathematics Figures Differently in Exact Solutions, Simulations, and Physical Models

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**Abstract** The role of mathematics in scientific practice is too readily relegated to that of formulating equations that model or describe what is being investigated, and then finding solutions to those equations. I survey the role of mathematics in: 1. Exact solutions of differential equations, especially conformal mapping; and 2. Simulations of solutions to differential equations via numerical methods and via agent-based models; and 3. The use of experimental models to solve equations (a) via physical analogies based on similarity of the form of the equations, such as Prandtl's soap-film method, and (b) the method of physically similar systems. Two major themes emerge: First, the role of mathematics in science is not well described by deduction from axioms, although it generally involves deductive reasoning. Creative leaps, the integration of experimental or observational evidence, synthesis of ideas from different areas of mathematics, and insight regarding analogous forms are required to find solutions to equations. Second, methods that involve mappings or transformations are in use in disparate contexts, from the purely mathematical context of conformal mapping where it is mathematical objects that are mapped, to the use of concrete physical experimental models, where one concrete thing is shown to correspond to another.

**Keywords** Equations · Models · Mathematics · Conformal mapping · Physically similar systems · Simulations

## 2.1 Introduction

The role of mathematics in scientific practice is too readily relegated to that of formulating equations that model or describe what is being investigated, and then finding solutions to those equations. That is a tidy but incomplete account of the role of mathematics in science. For one thing, mathematics is involved in experimental

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investigations to help answer investigative questions in ways other than finding solutions to mathematical equations. Further, the equations themselves can be involved in scientific practice in a variety of ways, some of which are substantively different from others. Identifying, clarifying, and explaining those different ways is my topic in this paper. To clarify the differences, I'll focus my discussion in this short paper on comparing how mathematics is involved in exact solutions, simulations, and experimental physical models.

## 2.2 Exact Solutions of (Differential) Equations

Which mathematical methods are used in solving equations of mathematical physics depends, of course, on the kind of equation. If we are talking about differential equations, then what is meant by a solution to the equation is a function that satisfies the equation. Sometimes the problem also specifies boundary values or initial values, adding more specific requirements that the function proposed as a solution to the differential equation must meet. Though it is possible to show, for some classes of equations, the existence and uniqueness of a solution, this is not the case in general. The Navier-Stokes equations in fluid dynamics are a set of partial differential equations, and the question of whether smooth, physically reasonable solutions to them exist is one of the Clay Mathematics Institute's Millenium problems.[1]

The study of solutions of partial differential equations, i.e., differential equations that involve partial derivatives, is an entire field of study unto itself. Partial differential equations are further classified into a variety of major types. The details of the classification are too involved to lay out here. For this discussion, I mention a few specific differential equations that have been given special names: the wave equation, the diffusion equation, and Laplace's equation. Many more could be mentioned.

Although the wave equation, the diffusion equation, and Laplace's equation are special cases of a differential equation, each constitutes an entire area of research in mathematical physics. Each applies to an indeterminately wide variety of phenomena in physics. The wave equation applies to electrical waves as well as mechanical waves and many other kinds of waves; the diffusion equation to heat diffusion (conduction), diffusion of particles, and diverse kinds of diffusion; and Laplace's equation likewise applies to a wide range of phenomena that arise in the study of heat, fluid flow, electrostatics, and similar phenomena in physics.

Our question in this context is thus how mathematics is involved in finding exact solutions of partial differential equations in mathematical physics. A straightforward approach to answering it is immediately thwarted, though, as there is no general method for finding exact solutions to partial differential equations. Deriving solutions to partial differential equations is not so much a matter of deductive logic, much less symbolic logic and set theory, as it is a matter of creativity by someone knowledgeable

[1] The official problem description is here: "Existence and Smoothness of the Navier-Stokes Equation" by Charles R Fefferman (https://www.claymath.org/sites/default/files/navierstokes.pdf).

in disparate fields of mathematics, some of which might be far afield from the equation in question. Often a significant amount of intellectual work is involved in identifying and reflecting on features of the equations to be solved, on boundary conditions, and on the difference that various kinds of boundary conditions make to the nature and existence of solutions, symmetries of a particular problem, and so on. Then, it is often a matter of resourcefulness. Even though I will later want to emphasize that there are other methods in mathematical physics than finding exact solutions, I still wish to register an appropriate appreciation of what is involved in finding exact solutions.

One of the most elegant and beautiful methods used in finding a function that is an exact solution to a partial differential equation is conformal mapping. The basic idea of the method is familiar from cartography: to map figures from the spherical earth onto a flat surface while at the same time preserving angles locally involves stretching and translation of the figures on the surface of a globe. The geometrical problem of which direction to head in to get from one point to another on the globe is solved by finding the line between those two points on the flat map, even though distances and areas are not correct on the flat map. In complex analysis, this basic idea was applied to map figures and graphs from one flat two-dimensional surface to another via transformations that involve stretching/shrinking and translation. Riemann's 1851 dissertation is credited with this creative suggestion (in spite of details about the proof and distinctions about the use of the term 'conformal' that would be brought up in discussing his formulation and proof today.) Ullrich notes:

> As an application of his [Riemann's] approach he gave a 'worked-out example', showing that two simply connected plane surfaces can always be made to correspond in such a way that each point of one corresponds continuously with its image in the other, and so that corresponding parts are 'similar in the small', or conformal . . . (Ullrich 2005, 454)

Conformal mapping was developed and used in complex analysis to find exact solutions to partial differential equations with great success in the 19th century. Bazant[2] remarks that

> The classical application of conformal mapping is to solve Laplace's equation:
>
> $$\nabla^2 \varphi = 0$$
>
> i.e. to determine harmonic functions in complicated planar domains by mapping to simple domains. The method relies on the conformal invariance of Eq. (1.1) [above], which remains the same after a conformal change of variables. (Bazant 2004, 1433)

Thus, problems involving complicated shapes of objects, boundaries, or surfaces would first be mapped, by a savvy choice of change of variables, to a new domain in which the shapes made the problem tractable (e.g., mapping an airfoil to a circle). Then, after solution in the new domain, the solved problem would be transformed back to the original domain, using the inverse of the function that had been used to map the problem from the original domain in which the shape was complicated to the

[2] I am indebted to Lydia Patton for introducing me to the work of Martin Z. Bazant, in her talk "Fishbones, Wheels, Eyes, and Butterflies", given at the Midwest Philosophy of Mathematics Workshop 17, University of Notre Dame, 12–13 November 2016.

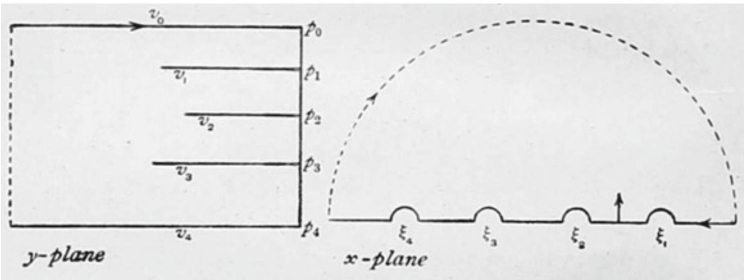


**Fig. 2.1** Schwarz-Cristoffel transformation of upper half-plane

domain in which the problem was tractable. Using the inverse function mapped the (now solved) problem from the domain in which it was solvable back to the original domain. This meant that the solution to the problem was mapped back to the original domain, too. So, it was a way of obtaining solutions to mathematical problems that were intractable as originally stated.

As mathematically elegant as this means of solving intractable problems was, it was not without a concrete precedent. Navigators in the 16th century could answer the question of which direction they should head on the spherical surface of the actual earth by consulting a flat map in which directions between points on the earth were preserved between corresponding points on the flat map—in spite of how distorted land shapes and area proportions were on the flat map. Likewise, a scientist could now solve problems that were otherwise intractable by working on the suitably transformed problem instead. The hard work is in finding the transformation that transforms the problem in such a suitable manner that the transformed problem is tractable and allows these two-way mappings. It is crucial to obtaining exact solutions to sets of differential equations via conformal mapping to be able to identify an appropriate transformation, i.e., an appropriate mapping function. Later in this paper, we will see other contexts in which attending to the role that an appropriate mapping function plays is likewise crucial, and why its role is so philosophically significant.

One of the earliest, and still well-known, transformations is the Schwarz-Cristoffel transformation. An example of the many problems to which it has been applied, taken from a nineteenth century text, is shown below.[3] While the figures are simple, it takes some time to contemplate and fully appreciate what the mapping effects, e.g., to understand just what the interior points of the polygon in Fig. 2.2 are mapped to.

The same mapping can be used to map all sorts of geometrical figures occurring in a variety of fields of physics. Sometimes a conformal transformation maps points at infinity to points on a figure. In fact, the Schwarz-Cristoffel style of map mentioned above is often illustrated by showing how the infinite (upper) half-plane can be mapped to various objects, such as a triangle. The visual insight as described in the 19th century text uses both Figs. 2.1 and 2.2 above: "when x turns to the right into

[3] Figures 2.1 and 2.2 are taken from Harkness and Morley 1898, 321).

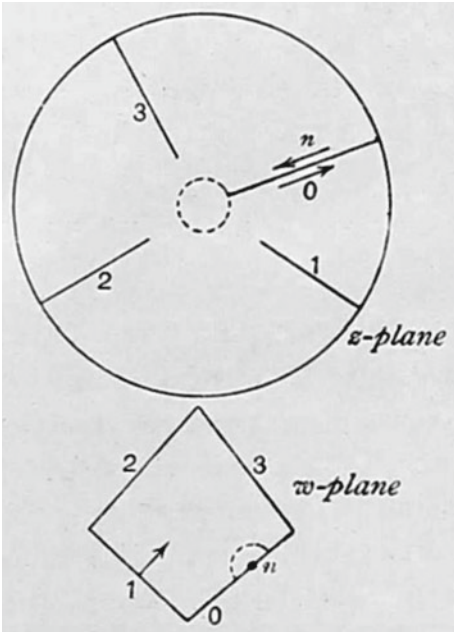


**Fig. 2.2** Schwarz-Cristoffel transformation: polygon



the upper half-plane, as indicated by the arrow in the right-hand part of Fig. 2.1, w turns also to the right into the interior of the polygon, as indicated by the arrow in the w-polygon of Fig. 2.2. Hence the upper half of the x-plane maps into the interior of the w-polygon" (Harkness and Morley 1898, 322).

Bazant comments that "important analytical solutions were thus obtained for electric  elds in capacitors, thermal fluxes around pipes, inviscid flows past airfoils, etc." citing twentieth-century works (Bazant 2004, 1434). Today there are computer programs that can carry out the transformations. These applications developed from the methods  rst that  rst arose in the mid-nineteenth century. However, the story of that method did not end once it achieved success in  nding exact solutions to problems expressible in the form of Laplace's equation. In the early twenty- rst century, the creativity and resourcefulness of some mathematicians led them to consider how this method of solution might be used elsewhere, i.e., for a different kind of problem in mathematical analysis. Though conformal mapping developed from within the study of Laplacian problems, it did not stay there. Bazant exuberantly reported in 2004 that:

> Currently in physics, a veritable renaissance in conformal mapping is centering around 'Laplacian-growth' phenomena, in which the motion of a free boundary is determined by the normal derivative of a harmonic function. . . . Such problems can be elegantly formulated in terms of time-dependent conformal maps, which generate the moving boundary from its initial position. (Bazant 2004, 1434)

Bazant describes developments that followed using iterated conformal maps, and then applied conformal mapping to Laplacian fractal-growth phenomena. His 2004 paper then brings the method to bear on a whole different class of problems in analysis: it proceeds from Laplacian fractal-growth phenomena to non-Laplacian fractal growth phenomena. It's the leap from Laplacian to non-Laplacian that's notable here: "One of our motivations here is to extend such powerful analytical methods to fractal growth phenomena limited by non-Laplacian transport processes. Compared with the vast literature on conformal mapping for Laplace's equation, the technique has scarcely been applied to any other equations." In fact, it just hadn't been thought possible to do so.

He explains his own thought process in applying the technique beyond Laplacian equations. Bazant says that, after some reflections on the fundamental features of Laplacian equations that lend themselves to solutions by conformal mapping, he questioned the common belief that the Laplacian operator is unique in being conformally invariant. He shows that certain systems of equations are conformally invariant, too. This then enables him to apply conformal mapping techniques to problems to which it had not been thought the technique of conformal mapping could be applied. As he tells the story, the advance boils down to one key insight, namely: "Everything in this paper follows from the simple observation that the advection operator transforms just like the Laplacian" (Bazant 2004, 1436).

Obviously coming up with these mathematical methods involves much more than deductive logic. What else? Analogy, knowledge of a variety of areas of mathematics, identifying and questioning a presumption prevailing among mathematicians, and creative leaps. So, it took more than the straightforward application of deduction in mathematics to come up with the methods Bazant reveals.

Once these methods are in hand, though, we can ask where and how mathematics is involved in the activity of using these methods to find exact solutions. Here we can be more precise about an answer: Not only is mathematics involved in putting the problem from physics into the mathematical language of differential equations, but it is involved in transforming that mathematical problem into one that is tractable. Once the problem is transformed, it can be solved in its novel form, and then one can use the inverse of the transformation process to put it back into the original domain, where it can be interpreted in the language of physics. Hence the mathematical functions that effect the mappings, and the construction of the domains where problems can be made tractable, are of crucial significance.

## 2.3 Simulations

Exact solutions of (partial differential) equations in mathematical physics are the exception rather than the rule, though.[4] Hence calculational methods have been developed. Numerical methods predate the current high-speed computational devices and

[4] This point is also made and discussed in McLarty, this volume Colin (2023).

systems in use today. It is often pointed out that some numerical methods associated with the calculus are due to, and hence in use as early as, Isaac Newton, but numerical methods are at least as old as papyri. New methods continue to be developed and enjoy widespread use, such as the finite element method in the latter twentieth century and agent-based modeling (cellular automata) in the twenty-first century, which will be discussed separately below. The widespread use of numerical methods and simulations does not, however, render exact solutions useless or unnecessary—when an exact solution does exist for certain cases, the exact solution serves the important role of a 'benchmark' against which numerical methods or simulations are compared.[5]

### *2.3.1 Simulations of (Differential) Equations of Mathematical Physics*

The simulations discussed here are numerical or other kinds of formal methods used in mathematical physics to yield approximations of solutions to the equations of mathematical physics. Whereas the methods used in finding exact solutions are elegant and deeply satisfying intellectually, the methods used in finding good simulations are nifty and deeply satisfying as well, in terms of their ability to provide practical and useful answers to analytically intractable problems. A wide variety of numerical methods have been developed, and the way that mathematics is involved in them varies accordingly. Some revolve around finite difference methods whereas others rely on probability, such as the Monte Carlo method. Perturbation theory and theory of errors are foundational theories for other numerical methods. Yet, with respect to the question in this paper, some generalizations can be made.

The ways that mathematics may be involved in the simulations under discussion here can be categorized in terms of three kinds of activity.

First, the development of equations, algorithms, and other formal methods in order to turn a problem involving the differential equation or systems of them into a problem that is well suited for computation.

Second, verification that the methods so developed will produce results that are solutions of the problem, within a certain band of error, and under certain limitations on the range of the variables in the problem. This step usually involves mathematical proofs and deductive reasoning, and often identifies the range of variable values over which the method is to be used.

Third, validation of the simulation via comparison with either exact solutions (used as 'benchmarks'), observational data, or experimental data, such as the results of an experimental physical model.

To be clear, these activities are not performed on an individual basis by an individual researcher very often anymore. Not only the first step of transforming the problem into one that is computationally tractable, but the steps of verification and validation,

[5] The use of benchmarks in simulations is likewise discussed in Patton, this volume Patton (2023).

are often carried out by communities of researchers (including researchers working for vendors of software) and inherited by subsequent researchers in the form of software. Hence it is quite common for someone to design and run a simulation without performing any of the three steps themselves. Ideally, such users would understand at least the basics of the first step (i.e., transforming the problem into one that is computationally tractable), and the second step, of understanding the ranges outside of which the method has not been verified. For, the users of the community-developed and verified software must make judgments in choosing which software to use for the problem they wish to solve, and in implementing the computationally tractable version of the problem in the software. Neither of these decisions is trivial, nor, if done well, free of mathematical reasoning. The third step, validation of the simulation, involves making comparisons between the values of quantities calculated by the simulation and values obtained some other way; either using an analytically closed form solution (exact solution) in the benchmark case, observational data, or experimental data from a specially constructed experimental physical model. Here the usual role that mathematics plays in working with measurements and uncertainties is involved.[6]

The topic of this paper is the role of mathematics in various methods in mathematical physics. However, the inclusion of the validation step hints at something else that is noteworthy. Computer software such as software that integrates fluid flow, heat conduction, and mass transfer in a computational flow dynamics program is not totally a matter of mathematics, even when numerical methods are included among mathematical methods. For empirical results of experimental studies are involved in the first step (development of formal methods that appropriately include physical factors), though sometimes in hidden ways (e.g., judgments as to whether a certain factor can be neglected), as well as in the third step (validation of the method/algorithm/software). If understood as a single linear three-step process, however, even if this point is appreciated, the description still does not quite reveal the extent to which experimental results are involved in the computation, for the process of building simulations involves much trial and error, iteration, and feedback—even in mathematical physics.

### 2.3.2 *Agent-Based Simulations*

The way that mathematics is involved in agent-based simulations is distinctive enough that they deserve separate mention. These kinds of simulations aren't implementations of algorithms to obtain approximations to solutions of equations. In agent-based simulations, rules are devised to proscribe the behavior of many individual agents acting in the same environment, in an attempt to model complex systems in which it is patterns of behavior that are of interest. Usually these rules for agents

[6] Relevant philosophical work on the topic of numerical methods and/or simulations can be found in Fillion (2017) and Lenhard (2019).



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are based in part on facts about other agents, such as how many agents of a certain kind are left, what they are doing, or how close other agents are to it. However, the rules are usually fairly simple rules mathematically speaking. So is the behavior of an individual agent: either ON or OFF (or ‘alive’ or ‘dead’). Different patterns arise depending on the initial configuration, and one soon begins to see the patterns of agent behavior as objects in their own right. They come to be regarded as agents and actions at a higher level than individual rule-following agents. Most philosophers were introduced to these ideas via the “Game of Life” by John H. Conway, often via Daniel Dennett’s 1991 “Real Patterns” (Dennett 1991, 27–51).

Agent-based simulations today are much more sophisticated. For instance, the environment in which the agents act can include resources that influence the agents’ capabilities to act, and the algorithms contain parameters that amplify or dampen rates or intensities. Swarm behavior of birds and fish, as well as the behavior of crowds and traffic, have been modeled with such agents. The kinds of uses researchers have made of one such simulation program alone—Uri Wilensky’s NETLOGO—is seemingly unlimited: art, biology, epidemiology, earth science, chemistry, hydrology, political science and social science, and so on. Hundreds of thousands of people from many different disciplines have used it.[7] Conway’s much simpler “Game of Life” can be programmed as a NETLOGO model, too (Wilensky 1998 and Wilensky 1999).

Since some phenomena that are described by differential equations, such as diffusion of particles and predator-prey interactions, can be modelled using agent-based models as well, the question of how the mathematics used in each are related naturally arises. NETLOGO has been expanded beyond agent-based models, to include a “Systems Dynamics” modeler, so that both the agent-based approach and the kind of approach used to develop differential equations to describe the same behavior can be taken. Wilensky describes the difference in how Wolf-Sheep predation is modeled when using the Systems Dynamics Modeler, versus the NETLOGO agent-based simulation modeler, as follows.

> System Dynamics is a type of modeling where you try to understand how things relate to one another. It is a little different from the agent-based approach we normally use in NetLogo models. With the agent-based approach we usually use in NetLogo, you program the behavior of individual agents and watch what emerges from their interaction. In a model of Wolf-Sheep Predation, for example, you provide rules for how wolves, sheep and grass interact with each other. When you run the simulation, you watch the emergent aggregate-level behavior: for example, how the populations of wolves and sheep change over time. With the System Dynamics Modeler, you don’t program the behavior of individual agents. Instead, you program how populations of agents behave as a whole.[8]

The way mathematics and logic are involved in the agent-based simulation, then, is really not just “a little” different from the way it is involved in the System Dynamics Modeler; it is strikingly and fundamentally different. For there is no equation describing the dynamics of the populations of predator and prey involved. Rather,

7 A running list of publications in which NetLogo was used or mentioned, many in scientific journals, is maintained on NetLogo’s website. The list contains hundreds if not thousands of papers from 1999 to the present, and more are added daily.

8 “NetLogo Systems Dynamic Guide”, n.p.

rules for individuals in the population are formulated, and in running the simulation, the dynamics of the predator and prey populations "emerges" from the agents acting according to those rules, as a matter of logical deduction.

However, models in use today are not always one or the other (i.e., not completely agent-based nor completely equation based); some are hybrid. NETLOGO is often integrated into other models, as part of a more comprehensive program. For instance, in hydrology, one model in use combines agent-based approaches in NETLOGO for the effect of agents who use water, along with finite-difference methods for solving equations of hydrological models of water flow (Castilla-Rho 2015). Further complicating any attempt at a strict taxonomy are newer developments in which the agents' behavior is continuous rather than discrete, as in the agent-based programs developed to model behavior of continua. The Turbulence model is one example (Wilensky 2003).; another is the vibration of a plate or membrane (Wilensky 1997).

When using agent-based models to investigate what emerges from agent-based approaches, apart from the aim of solving differential equations in mathematical physics, it could be that very little mathematics is involved, even when the behavior that emerges turns out to be a numerical or approximate solution to a differential equation. But in such a case, one is not looking for a solution to an equation, and thus there is no right or wrong in the matter. Inasmuch as the models are used to solve problems of mathematical physics, the process is broadly the same as the three part process described above for numerical methods: development of the formal method, verification of the formal method, and (depending on the aim of the modeler) validation of the formal method. Thus there is more involved than mathematics: observation and experimentation are involved, too.



## 2.4 Role of Mathematics in Experimental Physical Models

As mentioned earlier, although exact solutions have been found for a few special cases of the Navier-Stokes equations and other systems of partial differential equations, no general method for their solution is known. Conformal mapping is an elegant and powerful method, but it is not a general method; it relies on a researcher's creativity and resourcefulness. Observed phenomena related to turbulence, such as its onset and the separation of the laminar from turbulent regions of flow, are still not well understood nor mathematically tractable for many configurations. But methods of similarity in physics, which were used by Galileo and Newton in mechanics and dynamics (and likely before by others), were developed further for hydrodynamics in the centuries that followed. An examination of the reasoning used in problems in physics from pendula to vibration of plates, if traced back to their sources, would reveal the importance of using similarity along with observations and/or physical models to inform both the formulation of the differential equations and the methods of solving them.

Numerical methods and simulations are generally more cost-effective than building experimental physical models and setups for each and every situation one wishes to investigate, but the widespread use of numerical methods and simulations does not mean that they could be employed independently of the information gained from experimental physical models and setups. As there seems to be cultural amnesia about the significance of the role of experimental physical models and physically similar systems, at least among philosophers of science[9] and philosophers of mathematics, I ask the reader's indulgence here as I take the time to describe some early history of the topic that will be helpful in understanding the philosophical points in this paper.[10]

### *2.4.1 Physical Analogies in the Early Twentieth Century*

In his "Exact Solutions and Physical Analogies for Unidirectional Flows" Bazant (2016) notes that "In contrast to the more familiar case of Laplace's equation . . . conformal mapping cannot be as easily applied to Poisson's equation, since it is not conformally invariant." He notes, however, that "mathematical insights allowed Poisson's equation to be solved experimentally, long before it could be solved numerically on a computer." What can it mean to say that an equation can be solved experimentally? What kind of 'mathematical insights' could enable that?

The 'mathematical insights' Bazant names are "mathematical equivalence of beam torsion and pipe flow . . . [and] convective heat transfer"; and analogies with elastic membrane deflections, soap bubbles, and "the potential profile of electrically conducting sheets." From such insights, scientists were able to build a setup of one kind to determine the behavior—and for specific cases, determine values of quantities—of another kind. One of the earliest, most well-known, and tractable of these was the use of the analogy from membranes. A membrane was easy to create from soap film, hence it became known as "Prandtl's soap-film analogy." Prandtl's insight was that an analogy between two quite different phenomena could be made, "which could be described by the same differential equation if . . . specific parameters were replaced in each case by other [specific parameters.]" In Eckert's biography of Prandtl, he writes about Prandtl's paper describing analogous physical setups: "In the first case, the distortion of a soap membrane which is stretched over the opening of a container and bulges outward as a result of a small positive pressure in the container is considered; in the other, the twisting (torsion) of a bar that has the same diameter as the opening of the container" (Eckert 2019, 58). Eckert goes on to detail how the same differential equation describes two different kinds of quantities in the two quite different setups: "In the first case, the differential equation describes the lateral buckling as a result of the positive pressure in the container; in the second case,

9 Sherrilyn Roush is a rare exception. In "The Epistemic Superiority of Experiment to Simulation" (Roush 2018) she recognizes that "the solver" in a computer simulation incorporates sources other than 'the theory' (p. 4886).

10 A longer treatment is given in Sterrett, Susan G. "Physically Similar Systems: a history of the concept" Sterrett 2017.

the tension along the circumference of a bar cross-section induced by the twisting (torsion moment) of the bar” (Eckert 2019, 58).

Now, how, exactly, does one use soap lm to get the solution of a problem using a physical analogy? Here’s how: You would only have to construct the setup with the soap membrane stretched over the opening of a container. Then, you could take measurements as follows: “The angle of slope of the bulging membrane in the rst case corresponds to the shearing stress on the cross-sectional outline of the bar in the second case. The volume over the opening caused by the bulging of the soap membrane corresponds to the torsional stiffness of the bar” (Eckert 2019, 58). This is how problems can be solved by measurements of the membrane in the setup of the soap lm membrane—i.e., solved ‘experimentally’. But how is a set of measurements of distance in the soap lm informative about stress in a bar?

Here the form of the mathematical (partial) differential equations displays the analogy.

> *For the bar:* [T]he torsion of a bar along its long axis (x-axis in a cartesian coordinate system) is described by a stress function $y(y, z)$, . . . The stress function conforms to the equation
>
> $$\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 2G\theta$$
>
> where G is a material constant (torsion modulus) and $\theta$ the torsion per unit of length.
>
> *For the (soap-film) membrane:* A membrane that is stretched in the yz plane over an opening corresponding to the bar cross-section (tension S) and subjected on one side to a constant pressure p will bend towards the other side by an elongation $u(y, z)$. This elongation is described by the equation
>
> $$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{p}{S}$$
>
> From these two equations, *an analogy is produced between the stress function for the torsion of a bar and the bulging convexity of a membrane over an opening of the same surface as the bar cross-section*:
>
> $$\psi = \frac{2G\theta S}{p} u$$
>
> [11]

You cannot see or easily directly measure the stress in the bar but you can see—and measure—the distance that the soap membrane is bulging. So you construct the soap lm setup, measure the bulge $u$, and from it and the equation $\psi = \frac{2G\theta S}{p} u$ compute the stress in the bar.

The insight arises from the mathematical form, i.e., the analogy can be intuited because the physical quantities in both the bar and the membrane are expressed in terms of functions that are solutions to partial differential equations—not due to any experiential familiarity or knowledge about torsional stress in bars or deflections in thin membranes. Prandtl felt the approach could be used in many more elds.

[11] Eckert (2019), 59.



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To facilitate recognition of analogous physical situations by looking at the mathematical equations, he felt it was of utmost importance to standardize mathematical expressions across different scienti c and technical areas of science and engineering. The idea was that doing so would increase the opportunities for the kind of mathematical insights he had with the soap membrane analogy. The reason it was so important to facilitate such insights was that seeing such analogies would allow people to obtain solutions to partial differential equations not available any other way. I take it that this is just what Bazant was referring to in saying that “mathematical insights allowed Poisson’s equation to be solved experimentally, long before it could be solved numerically on a computer” (Bazant 2016, 024001–2).

Bazant’s 2016 paper on physical analogies published in *Physical Review Fluids* shows how really fruitful this approach is, even today. He writes about the “common mathematical problem [that] involves Poisson’s equation from electrostatics

$$-\nabla^2 u = k$$

typically with constant forcing k and Dirichlet (no-slip) boundary conditions on a two-dimensional domain”, and notes that “The same problem arises in solid mechanics for beam torsion and bending” and, in fact, in two dimensions, arises “for a remarkable variety of physical phenomena” (Bazant 2016, 024001–2). He provides a survey and then expands the number of physical analogies even farther. He lists a total of seventeen, sketched in a gure; the caption lists them as follows:

> Seventeen analogous physical phenomena from six broad elds, all described by Poisson’s equation in two dimensions. Fluid mechanics: (a) Poiseuille flow in a pipe, (b) circulating flow in a tube of constant vorticity, and (c) groundwater flow fed by precipitation. Solid mechanics: (d) torsion or (e) bending of an elastic beam, and (f) deflection of a membrane, meniscus, or soap bubble. Heat and mass transfer: (g) resistive heating of an electrical wire, (h) viscous dissipation in pipe flow, and (i) reaction-diffusion process in a catalyst rod. Stochastic processes: (j) rst-passage time in two dimensions, (k) the chain length pro le of a grafted polymer in a tube, and (l) the mean rate of a diffusion-controlled reaction. Electromagnetism: (m) vector potential for magnetic induction in a shielded electrical wire, and the electrostatic potential in (n) a charged cylinder or (o) a conducting sheet or porous electrode. Electrokinetic phenomena: (p) electro-osmotic flow and (q) streaming current in a pore or nanochannel.

One of them is especially surprising, and shows the creativity and intellectual insight in recognizing these analogies: the stochastic processes (j, k, and l).

The role mathematics plays here is quite explicit: rst, an understanding of analysis as used in mathematical physics allows someone to formulate partial differential equations in a canonical form; second, comparison of partial differential equations from various parts of mathematical physics provides opportunities to recognize analogies between very different areas of mathematical physics; and third, once an analogy is recognized, the equation permits someone to use a setup analogous to the one that one wishes to have a solution of, to obtain a solution. The mathematical equation also provides the mapping from a measured quantity in one of the setups to the inferred quantity in the other.



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There are philosophers of science who are familiar with this kind of physical analogy, in a general way. A common example cited is the harmonic oscillator, which is the linear differential equation for a mass on a spring—and for many other physical systems in nature, as well. Francisco Guala and Chris Pincock's works, to take two recent examples, exhibit familiarity with cases where an equation is instantiated by more than one situation, and Pincock speci cally mentions partial differential equations, including Laplace's equation and Poisson's equation. Pincock briefly discusses cases of scale similarity and dynamical similarity (experimental scale models); he assumes that the dimensionless parameters used to effect this kind of similarity are obtained from the equations that describe the two similar situations.[12] We shall see that although this may often be so, it is not necessarily the case: there is another basis for similarity that does not require knowledge of even the governing equations. The explanation for how we can establish that kind of similarity without knowledge of the governing equation requires looking more broadly than either of these two philosophers have, into the foundations of metrology and the relationship between different kinds of mathematical equations and how they are related to physical quantities. We turn now to that method: the method of physically similar systems, via the use of dimensional equations.



### *2.4.2 The Method of Physically Similar Systems*

The use of similarity by European scientists in the nineteenth and early twentieth century was wide-ranging if not ubiquitous. There is not room here to convey the breadth and depth of the uses made of similarity in physics, but I have tried to do so elsewhere, and I refer the interested reader to that paper, "Physically Similar Systems: A history of the concept" (Sterrett 2017). For the question that is the focus of this paper, the role of mathematics in experimental physical models, I pick out a few exceptional papers to highlight the distinctiveness of the method.

Against a backdrop of the impressive and exciting accomplishments made by reasoning from analogy, resulting from the ability to formulate so many different areas of physics in terms of partial differential equations of the same form, Helmholtz brings a critical attitude to bear. He points out that there is sometimes a difference in the behavior of two situations that are described by the same partial differential equations—including the same boundary conditions. The two situations to which he draws attention are: "the interior of an incompressible fluid that is not subject to friction and whose particles have no motion of rotation" and "stationary currents of electricity or heat in conductors of uniform conductivity" (Helmholtz 1891a, 58). These two con gurations share the same formulation in analysis, i.e., "precisely the same" partial differential equations, and they have the same boundary conditions. Yet, their behaviors differ. Helmholtz considers, and dismisses as implausible, the



[12] Pincock (2012). Also all the people who have mentioned the harmonic oscillator and its several instantiations have done so, of course.

explanation that the difference is a matter of the hydrodynamical equations being an "imperfect approximation to reality" (Sterrett 2017, 392). Rather, he says, apparent contradictions between the hydrodynamic equations and "observed reality" disappear once it is recognized that discontinuous motions can occur in fluids. This is not a case of the hydrodynamic equations being wrong, though. As I put it in explaining Helmholtz's view in previous work: "The problem with the hydrodynamic equations is not that are wrong, for they are not; they are 'the exact expressions of the laws controlling the motions of fluids.' The problem is that 'it is only for a relatively few and specially simple experimental cases that we are able to deduce from these differential equations the corresponding integrals appropriate to the conditions of the given special cases.' So, the hydrodynamic equations are impeccable; it's their solution that is the problem" (Sterrett 2017, 392–3; citations from Helmholtz 1891a).

In case this seems puzzling, recall how the solution and equations they are a solution to are related. The hydrodynamic equations are governing hydrodynamic equations, but when it comes to the solution—and here he is talking about an exact solution—the solution may be expressed in terms of an equation that involves an integral. The function that is the exact solution is a function satisfying that integral equation. And evaluating that integral is where attentiveness to discontinuities in the fluid is called for, as it involves considering the range over which the pressure at every point varies. Helmholtz next considers a suggestion to simplify the problem. But he rejects that, too, as in some cases "the nature of the problem is such that the internal friction [viscosity] and the formation of surfaces of discontinuity cannot be neglected" (Helmholtz 1891b, 67). So you don't want to deal with the problem by simplifying it in a way that writes that complexity of the picture.

Another way to understand Helmholtz's point here is to consider that people often used these analogies to find solutions to equations in the way we discussed Prandtl doing above using the soap bubble membrane. For instance, someone might use an electrical circuit or setup to find the solution in a fluid flow setup. Then, Helmholtz's point would be that even though the governing partial differential equations are the same, and the boundary conditions are the same, discontinuities in fluid flow can arise in the fluid setup that will not arise in the electrical setup. The numerical value of the pressure in the fluid flow setup turns negative in the interior of the fluid, and the flow separates. Today in a practical context people would say that there is cavitation in the flow, or that the flow cavitates, as flow discontinuities tend to form.

As I explained in earlier work (Sterrett 2017, 391), the surfaces of discontinuity Helmholtz identified are an obstacle to finding a solution, too. For, as Helmholtz writes, "The discontinuous surfaces are extremely variable, since they possess a sort of unstable equilibrium, and with every disturbance in the whirl they strive to unroll themselves; this circumstance makes their theoretical treatment very difficult." Theory being of very little use in prediction here, he says, "we are thrown almost entirely back upon experimental trials, . . . as to the result of new modifications of our hydraulic machines, aqueducts, or propelling apparatus" (Helmholtz 1891b, 67). Well, what sort of experimental trials can he mean, if not the kind of analogy that he has just explained cannot be relied upon?



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Helmholtz says there is another method, which he describes as follows: “In this state of affairs [the insolubility of the hydrodynamic equations for many cases of interest] I desire to call attention to an application of the hydro-dynamic equations that allows one to transfer the results of observations made upon any fluid and with an apparatus of given dimensions and velocity over to a geometrically similar mass of another fluid and to apparatus of other magnitudes and to other velocities of motion” (Helmholtz 1891b, 68). Gabriel Stokes had already, in 1850, spoken of ‘similar systems’ and identified conditions under which one could make inferences about similar motions and about the relation of forces in similar systems; these conditions have to do with relations between quantities in the systems (Stokes 1850, Sect. 5). In later reviews of similarity in hydrodynamics, Helmholtz’s and Stokes’ methods are identified as the same method, so Helmholtz is likely drawing on this earlier 1850 work of Gabriel Stokes, the same Stokes for whom the Navier-Stokes equations are named.

The method Helmholtz means here is not a matter of deduction from theory, or even of finding a solution to equations. In this paper of 1873, which soon become foundational in empirical methods in flight research, we see that theory is still involved in the kind of inference he describes. However, the way that theory is involved is to allow someone to “transfer the results of observations made on one thing (system, machine, mass of fluid, apparatus) over to another thing (system, machine, mass of fluid, apparatus)” (Sterrett 2017, 68). It is implied, I think, that the reason this is helpful in making predictions is that some observations are more accessible, and some things are easier to manipulate and take measurements on, than others. This is reminiscent of the approach used in the realm of applied mathematics when using conformal mapping to obtain exact solutions to partial differential equations, i.e., to first transform a problem to a domain where it becomes more tractable, solve the problem in the new domain, and then transfer the solution back to the original problem. Now Helmholtz is talking about doing this with concrete, physical things, but not based on the fact that both are instantiations of the same differential equation, which, he has just shown, is not sufficient to allow one to transfer results from one setup to another. Thus, while there is an apparent similarity, Helmholtz’s reasoning does not have exactly the same basis as Prandtl’s soap-film method.

Helmholtz shows how one can use the governing hydrodynamic equation to which one does not have a solution to construct a mapping between two different fluids that may have different characteristics. The constraint that both of them must satisfy the hydrodynamic equations is used to determine how the various geometrical and non-geometrical quantities (time, fluid density, pressure, and coefficient of friction or viscosity) must be related. This induces a mapping (via a change of variables, as in conformal mapping) between the two fluid masses. He also distinguishes compressible from incompressible, and cohesive (liquids) from noncohesive (gases), and so on, to determine all the constants used in the change of variables that induces the mapping. In that paper, he shows how one can compare “a mass of water in which a ship is situated” and “a mass of air in which an air balloon is situated” (Helmholtz 1891b, 73). This process does use the governing hydrodynamic equation to guide the construction of the mapping, but it also formalizes the "peculiarities of air and water" in doing so, too. His approach attends to how quantities are related to each other.

In the 1873 paper, Helmholtz identifies a number of dimensionless ratios, each of which is to this day considered fundamental in establishing similarity in meterology and fluid dynamics. Dimensionless ratios are not constants.[13] Dimensionless ratios can take on various numerical values, and because the values of these ratios are informative about the thing they describe, they are often called dimensionless parameters. (A very simple case is the Mach number, which is the ratio of two velocities, hence dimensionless. Everyone is familiar with the Mach number being used to indicate whether flight is subsonic or supersonic, for instance.) Thus dimensionless parameters are informative about the similarity of two things with respect to that physical feature, and they are used to judge whether two things are similar and the ways in which they are similar. Helmholtz does not elaborate much on how one is to determine exactly what is being compared; here he uses the terms "mass of water" and "mass of air." A more general formulation of Stokes' earlier paper and Helmholtz' insight here was enabled in the early years of the twentieth century, as the field of thermodynamics developed and the notion of a system in thermodynamics (conceived of as subsuming mechanics within it) was developed.

Osborne Reynolds, Ludwig Prandtl, and Rayleigh each individually made important contributions regarding similarity in hydrodynamics worthy of in-depth discussion in their own right, and I have discussed them in a longer historical paper on the subject (Sterrett 2017, 394–397). In this chapter, we skip over them to get to the definitive statement of physically similar systems, which came from a thermodynamicist who was working as a physicist at the National Bureau of Standards: Edgar Buckingham. Though an American, Buckingham had travelled to Germany for his PhD work, working with Wilhelm Ostwald in Leipzig on a dissertation on thermodynamics. He modestly described his contribution as merely attempting to state the methods in use by researchers who used similarity methods, and to identify a rigorous basis for them, but his statement in terms of "physically similar systems" and "dimensional equations" was distinctively different from their works, and is considered the landmark work today.[14]

Buckingham took a more formal approach, one that was rooted in the nature of scientific equations: the requirement of dimensional homogeneity. It is really about the logic of equations. He was not the first to do so: Joseph Bertrand had likewise located the foundations of similarity for both mechanics and hydrodynamics in the

[13] I mention this because I have found that, inexplicably, many philosophers think they are.

[14] Philosophers may be familiar with Buckingham's work on dimensional analysis via Percy Williams Bridgman's book *Dimensional Analysis* (Bridgman 1922). Few if any have noticed Bridgman's note in the Preface to that book expressing his indebtedness to the papers of Buckingham and to Hersey at the Bureau of Standards for presenting Buckingham's results in a series of lectures. In my entry "Dimensions" in the *Routledge Companion to the Philosophy of Physics* (Sterrett 2021), I compare their treatments, how Bridgman's treatment follows along the lines of Buckingham's, yet what has been lost in Bridgman's partial understanding of Buckingham's very deep and philosophical work on the logic of dimensions.

principle of the homogeneity of equations, and attributed the insight to Isaac Newton (Bertrand 1878; Bertrand 1847). Newton had written about dimensions and units and their relation to similarity of systems, even using the term "similar systems."

In the now-landmark 1914 paper "On Physically Similar Systems: Illustrations of the Use of Dimensional Equations", Buckingham's starting point is "the most general form of a physical equation." What he means by a physical equation is an equation that describes "a relation which subsists among a number of physical quantities of n different kinds." Quantities, not variables. Dimensions, as the term is used in dimensional analysis, was developed in the context of foundational investigations into relations between quantities (e.g., Newton in investigating mechanics; Fourier in investigating heat, 19th century physicists on the relations of quantities in electromagnetism).

To get at the logic of the form of an equation that expresses a relation between different kinds of quantities, Buckingham then pares down the number of quantities by consolidating quantities of the same kind: "If several quantities of any one kind are involved in the relation, let them be speci ed by the value of any one and the ratios of the others to this one" (Buckingham 1914a; 345). Then, to start off simply, he restricts the discussion to cases where those ratios do not change over the course of time being considered. We are left with an equation expressing the relation between n different kinds of quantities of the form $F(Q_1, Q_2, ...Q_3) = 0$, where F is an unde ned function of quantities. Further reasoning about equations in physics leads to the conclusion that every 'complete' equation of physics can be expressed in the form:[15]

$$\sum M Q_1 b_1 \quad Q_2 b_2 ... Q_n b_n = 0$$

This is where the logic of equations of physics comes in, as this is where a principle concerning constraints on the equations of physics, i.e., the principle of dimensional homogeneity, comes in. I rst give this intuitive sense of the principle: in an equation of physics, only commensurable quantities may be equated; only commensurable quantities may be added. Buckingham refers to it as "a familiar principle", credits Fourier with rst stating it, and states it in his paper as follows: "all the terms of a physical equation must have the same dimensions" or, alternatively, "every correct physical equation must be dimensionally homogeneous" (Buckingham 1914a, 346).

Some ratios, such as $LT^{-1}$ (length divided by time), will have a dimension, whereas others, such as Mach number, which is the ratio of the speed of a projectile to celerity (the speed of sound in the medium in which it is traveling) will not, since the dimension is $LT^{-1}L^{-1}T$. By the time Buckingham was writing, there were already well-known dimensionless ratios such as Mach number. These are parameters, not constants. They can take on many values, and the value they take on is often very informative (e.g., as Mach number varies from less than 1, to 1, to larger than 1, it indicates a change from subsonic to critical point to supersonic flight). Reynolds number (density velocity length divided by dynamic viscosity)

[15] See Buckingham (1914a), 346 for his explanation of what the exponents in this equation indicate.

is likewise dimensionless and informative. In this case, it is indicative of flow regime as it proceeds from laminar to turbulent flow. In his 1914 paper, Buckingham goes on to show that, from his starting point of the most general form of a physical equation, he can derive the fact that every physical equation can be expressed in terms of dimensionless parameters, i.e., in the form $\psi(\pi_1, \pi_2, \pi_n) = 0$, where $\psi$ is an unknown function and the dimensionless parameters $\pi_n$ are independent of each other. I take the latter to mean that none of the dimensionless parameters $\pi_n$ in the equation can be expressed in terms of the others (Buckingham 1914a, 347).

In a brief work reporting on his progress on the topic of physically similar systems for the first time (May 1914), Buckingham deduces the following, presenting it as a theorem about scientific equations:

> The theorem may be stated as follows: If a relation subsists among a number of physical quantities, and if we form all the possible independent dimensionless products of powers of those quantities, any equation which describes the relation is reducible to the statement that some unknown function of these dimensionless products, taken as independent arguments, must vanish. (Buckingham 1914b, 336)

I've provided the expository discussion above to help make some sense of what he says here, but for the purposes of this paper I also wish to emphasize that it is a theorem about the equations of physics, where physics is taken in a very inclusive sense. Dimensions can loosely be thought of as kinds of quantities for our purposes here.[16]

In later correspondence (to Rayleigh), Buckingham explains the role of logic and algebra as compared to the role of physical theory in his account of physically similar systems:

> I had therefore . . . to write an elementary textbook on the subject for my own information. My object has been to reduce the method to a mere algebraic routine of general applicability, making it clear that Physics came in only at the start in deciding what variables should be considered, and that the rest was a necessary consequence of the physical knowledge used at the beginning; thus distinguishing sharply between what was assumed, either hypothetically or from observation, and what was mere logic and therefore certain.[17]

Now, being able to express a physical equation as an *undetermined* function of dimensionless parameters is extremely empowering in terms of establishing similarity. In this discussion, I am interested in concrete physical models, but the use of similarity is not restricted to concrete physical models. The concept of physically similar systems can be applied to anything in physics that can be characterized as a system in the sense the term is used in thermodynamics (which includes all of classical mechanics, for classical mechanics is thermodynamics without consideration of the role of heat).

The methodology of physically similar systems enables one to use a physical model to investigate phenomena in another system, but not, as in Prandtl's use of analogy, by insight into the form of the equation describing the system behavior—and that's what is so remarkable about the method of physically similar systems. The

[16] I provide a more rigorous discussion in "Dimensions" (Sterrett 2021).

[17] Edgar Buckingham: Letter to Lord Rayleigh (John William Strutt) dated November 13, 1915, handwritten on official National Bureau of Standards stationery.

process proceeds as follows: First, it is established that the model and what it models are physically similar systems (with respect to some relation). Usually a system $S'$ is constructed in such a way as to be similar to the system S, and is regarded as an experimental model of it, whether S exists in the actual world or is only a design on paper. This is done by constructing the model and setting conditions so that the values taken on by the dimensionless parameters (e.g., Mach number, Reynolds number) are the same in the model as in what it is modeling. Buckingham's discussion, while somewhat formal, provides the basis for this: "Let S be a physical system, and let a relation subsist among a number of quantities which pertain to S. Let us imagine S to be transformed into another system $S'$ so that $S'$ 'corresponds' to S as regards the essential quantities." He eventually deduces the nature of the similarity transformation, spelling out how one would go about setting values so that the values of the $\pi$s are the same in S as in $S'$ (Buckingham 1914b, 353ff).

The point is elegant, reminiscent of the elegance of conformal theory: the constraint that must be satisfied in constructing the system $S'$ is just that the value of the dimensionless parameters that appear in the general form of the equation—the arguments of the function $\phi$—are the same in $S'$ as in S. Thus, the approach Buckingham takes in constructing similar systems, the foundational basis for the construction of physically similar systems, is not a method peculiar to any particular part of physics. In that paper, he goes on to discuss applications of the method to electromagnetism (energy density of an electromagnetic field, radiation from a moving electron, and others), dynamics, and heat convection, and argues that the method is quite general. This can be very puzzling, for it doesn't seem that there is enough information in the antecedent of the theorem to permit the conclusion. Is there something about scientific equations that contributes to the argument? The answer to this is yes, and it is a matter of metrology, the science of measurement.

The requirement of a coherent system of measurement, i.e., one in which the relations between the units are the same as the relations between quantities, was adopted in the nineteenth century, and by the time Buckingham was thinking through the basis for similarity while working at the National Bureau of Standards, he could take coherence of the system of units being used in physics for granted. The logic of the quantities on which measurement systems are based is actually logically prior to the measurement system, so there is a lot of physics built into the system of measurement. The requirement that the system of units used in physics be coherent (in the above sense of the term) thus allows logical consequences to be drawn that could not be otherwise be drawn.[18]

I think it clear that, despite the spare elegance of Buckingham's account of physically similar systems, there is more at work than mathematics in accounting for the power of physically similar systems. The method he described for model experiments (experimental physical models) was based on a formalism that is in some sense even more fundamental than the mathematical equations describing the behavior of interest, and yet in some sense dependent on them: dimensional analysis, which we can think of as a formalism or language for quantities and the relations between them.

[18] I discuss this in more detail in Sterrett (2019).

Metrology and systems of measurement were developed in tandem with new developments in physics, and the use of the kinds of equations used in modern physics (as opposed to the proportional equations of previous eras) created the need for them. They were developed in order to enable the use of numerical interpretations of equations of physics.[19] That knowing the answer to "What are the relevant quantities involved?" is enough is striking, and is often met with incredulity. What was remarkable about Prandtl's soap- lm method was that the solution to an equation could be obtained experimentally using analogy between equations. But the method of physically similar systems goes one step farther, in that one need not even have the equation describing the phenomenon of interest in hand. It is understandable that the claim is met with incredulity, unless and until the role of the coherence of a system of units for physics is appreciated.

I have two comments regarding the point that one can construct experimental physical models to investigate a phenomenon even when one does not have in hand an equation describing the phenomenon of interest.

First, the point is limited to physics (as opposed to areas of biology or sociology where one cannot draw on the same features of a system of measurement). In physics, knowing which of the quantities are relevant to a phenomenon of interest—and which are not—is actually quite a good deal of information. This is of course due to the role of the coherence of the system of units used in physics.

Second, the formulation that Buckingham provided is really very special and, I think, contingently available to us. The point about being able to do without the equation describing the phenomenon to be modeled might not have been recognized were it not for his imprint on the method.[20] The proofs and practices behind model experiments were actually not entirely new in 1914, a point he freely offered himself. But, the approach in terms of an investigation into the "most general form of physical equations"? That was new. The attention to the nature and role of the equations of physics—that attentiveness came from a physicist who had been in a community of philosophically engaged physicists who were actively discussing what units (e.g., temperature, charge) were needed in physics and what the logic of numerical scienti c equations was. He had been in the thick of discussions about the role of equations in physics while studying for his doctorate in Germany, when Ostwald and Boltzmann were in dialogue. The question of whether equations were indispensable in physics, or whether, alternatively, models and analogies might do that work for the emerging physics of the day, was seriously debated (e.g., by Ludwig Boltzmann). And, then, years later, he found himself assigned the question of whether there was a proper methodology for the interpretation of model experiments, this time in the milieu of the National Bureau of Standards in Washington DC, where scienti c research into

[19] As an historical-philosophical account of equations in physics and the concomitant development of metrology, I recommend De Courtenay (2015).

[20] Note that Helmholtz, writing much earlier and in an era that predated coherence of a system of units for physics in the sense it is used here, was able to show how to establish the similarity of two physical situations, too, but that he did so by using the hydrodynamical equations. He derived dimensionless forms of the equations, and then established that if the dimensionless coef cients were the same between two situations, they would have similar motions.

establishing standards for units was being done. That he begins that investigation with the topic of “the most general form of physical equations” is something I  nd noteworthy as a philosopher.

It would be a mistake to dismiss Buckingham’s work, as so many philosophers have, as about “little scale models” or about engineering technology.[21] It is some of the deepest thinking about the logic of the equations of physics there is. Yes, of course, it was possible only due to all the work on similarity by other physicists, and there is no doubt that he was fortuitously located in a position unique to those writing about the basis for model experiments (experimental physical models). What should be recognized is how much more enlightened we are—or at least, could be—about the nature and role of equations, as a result of this philosophically informed account of the basis for model experiments (experimental physical models). The method of physically similar systems is not restricted to scale models, either, but is generally applicable.

## 2.5 Conclusion

In none of the uses of mathematics surveyed in this paper—exact solutions, simulations (numerical approximations and agent-based) and experimental physical models—is the solution to an equation simply a matter of deduction. Even in the purest example of mathematics surveyed here, i.e., exact solution of partial differential equations, the role of insight was crucial. Conceiving of the kind of mapping that might work for the problem at hand is a far cry from a straightforward application of deductive methods. This was no less true for simulations. In addition, with numerical simulations, we saw that experiential information was inextricably knit into the process by which computer simulations are produced.

We also encountered practices in science in which results that previously were thought to require an equation describing the phenomenon of interest were obtained without use of the equation. Though not news, it should give us pause that agent-based models (consisting of many agents with very simple rules) are being used to investigate behavior previously investigated using differential equations describing the behavior of continua. Most numerical methods employ a differential equation or equations describing the target behavior in some way, but agent-based models are completely different in this regard. The use of physically similar systems is another scienti c practice wherein results previously thought to require having an equation describing the phenomenon of interest were obtained without the equation. Nineteenth century methods for using concrete physical models based on insights about analogies between equations were developed by Prandtl, Stokes, Helmholtz, and many others. But their similarity methods, while somewhat reliant on insight about mathematical analogies, still centered on the differential equations governing

[21] One prominent philosopher of science, referring to Buckingham, chastised me for writing about the work of “an obscure engineer” in my book *Wittgenstein Flies A Kite* (Sterrett 2005).

the phenomenon or behavior to be investigated. The method of physically similar systems does not. In fact, it does not require having the equation in hand in order to construct and use model experiments.

Surprisingly, it was in what one might have thought the application most dependent on practical insights, i.e., using concrete physical models, that we came across something closest to a general method for  nding a solution. While it is certainly true that experiential knowledge is involved in various ways in using the method of physically similar systems, some of them nontransparently so, it is notable that the mappings (between what is to be modeled and the model, and then from model results back to what is to be modeled) can be obtained without having the equation in hand. A set of relevant dimensionless parameters can be obtained from partial knowledge, i.e., from less knowledge than the equations describing the phenomenon being investigated.

It was in examining the basis for physically similar systems that an account explaining why experimental physical models can be so informative about what they model was provided. In fact, Buckingham discussed equations of physics using a whole other formalism: the language of quantity, e.g., of dimensional analysis. He used the formalism of a different kind of equation, dimensional equations, in tandem with the kinds of equations used in physics. The explanation of why the method of physically similar systems worked as well as it did, when it did, had to do with something not contained in the practices of deriving solutions either computationally or via mathematical proofs. It relied on the coherence of the system of units used in physics, which is not a matter of mathematics or logic, but is constructed apart from it, and involves both empirical results and community decisions (Sterrett 2019; De Courtenay 2015; De Courtenay 2021). It is a vast understatement to say that this point is not appreciated in philosophy of science or philosophy of mathematics. It has not gone totally unrecognized, but the work on it is seldom taken up in discussions where it would shed much light.[22]

Though this short paper is an investigation into the role of mathematics in science, it began and ended discussing equations. It has ended by recognizing a much more complex account of the equations of physics than occurred at the start (when considering exact solutions to differential equations). Because of the work on the role of dimensional equations (Buckingham 1914a; Buckingham 1914b) in showing how transformations that could solve questions about the behavior of physical systems could be answered in spite of not having an equation for that behavior, and the role of metrology (De Courtenay 2015) in enabling the use of the kinds of equations now used in physics, we can now see what we might not have realized otherwise about equations: that there is much more to them than what they say.





[22] De Courtenay (2015) provides an excellent account that appreciates that the role of metrology in enabling the use of numerical equations in science is well-hidden (as intended).

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